CSE 116: Fall 2019

Introduction to Functional Programming

Formalizing Nano

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Formalizing Nano

Goal: we want to guarantee properties about programs, such as:

- evaluation is deterministic
- all programs terminate
- certain programs never fail at run time
- etc.

To prove theorems about programs we first need to define formally

- their *syntax* (what programs look like)
- their *semantics* (what it means to run a program)

Let's start with Nano1 (Nano w/o functions) and prove some stuff!

Nano1: Syntax

We need to define the syntax for *expressions* (*terms*) and *values* using a grammar:

```
e ::= n | x                -- expressions
    | e1 + e2
    | let x = e1 in e2

v ::= n                    -- values
```

where $n \in \mathbb{N}, x \in Var$

Nano1: Operational Semantics

Operational semantics defines how to execute a program step by step

Let's define a *step relation* (*reduction relation*) `e => e'`

- "expression `e` makes a step (reduces in one step) to an expression `e'`

Nano1: Operational Semantics

We define the step relation *inductively* through a set of *rules*:

```
               e1 => e1'        -- premise
[Add-L]   --------------------
          e1 + e2 => e1' + e2   -- conclusion

              e2 => e2'
[Add-R]   --------------------
          n1 + e2 => n1 + e2'

[Add]     n1 + n2 => n               where n == n1 + n2

                        e1 => e1'
[Let-Def] --------------------------------------
          let x = e1 in e2 => let x = e1' in e2

[Let]     let x = v in e2 => e2[x := v]
```


Nano1: Operational Semantics

Here `e[x := v]` is a value substitution:

```
x[x := v]                  = v
y[x := v]                  = y        -- assuming x /= y
n[x := v]                  = n
(e1 + e2)[x := v]          = e1[x := v] + e2[x := v]
(let x = e1 in e2)[x := v] = let x = e1[x := v] in e2
(let y = e1 in e2)[x := v] = let y = e1[x := v] in
e2[x := v]
```

Do not have to worry about capture, because `v` is a value (has no free variables!)

Nano1: Operational Semantics

A reduction is *valid* if we can build its **derivation** by “stacking” the rules:

```
    [Add] --------------------
              1 + 2 => 3
[Add-L] -----------------------
        (1 + 2) + 5  =>  3 + 5
```

Do we have rules for all kinds of expressions?

Nano1: Operational Semantics

We define the step relation *inductively* through a set of *rules*:

```
               e1 => e1'        -- premise
[Add-L]    --------------------
           e1 + e2 => e1' + e2   -- conclusion

              e2 => e2'
[Add-R]    --------------------
           n1 + e2 => n1 + e2'

[Add]      n1 + n2 => n              where n == n1 + n2

                     e1 => e1'
[Let-Def]  --------------------------------------
           let x = e1 in e2 => let x = e1' in e2

[Let]      let x = v in e2 => e2[x := v]
```


1. Normal forms

There are no reduction rules for:

- n
- x

Both of these expressions are *normal forms* (cannot be further reduced), however:

- n is a *value*
  - intuitively, corresponds to successful evaluation
- x is *not* a value
  - intuitively, corresponds to a run-time error!
  - we say the program x is **stuck**

2. Evaluation order

In `e1 + e2`, which side should we evaluate first?

In other words, which one of these reductions is valid (or both)?

1. `(1 + 2) + (4 + 5) => 3 + (4 + 5)`
2. `(1 + 2) + (4 + 5) => (1 + 2) + 9`

Reduction (1) is *valid* because we can build a **derivation** using the rules:

```
          [Add] ----------
                  1 + 2 => 3
[Add-L] ----------------------------------
        (1 + 2) + (4 + 5)  =>  3 + (4 + 5)
```

Reduction (2) is *invalid* because we cannot build a derivation:

- there is *no rule* whose conclusion matches this reduction!

```
                ???
[???] -----------------------------------
      (1 + 2) + (4 + 5)  =>  (1 + 2) + 9
```


Evaluation relation

Like in λ-calculus, we define the **multi-step reduction** relation `e =*> e'`:

`e =*> e'` iff there exists a sequence of expressions `e1, ..., en` such that

- `e = e1`
- `en = e'`
- `ei => e(i+1)` for each `i in [0..n)`

Example:

```
    (1 + 2) + (4 + 5)
=*> 3 + 9
```

because

```
   (1 + 2) + (4 + 5)
=> 3       + (4 + 5)
=> 3       + 9
```


Evaluation relation

Now we define the **evaluation relation** `e =~> e'`:

`e =~> e'` iff

- `e =*> e'`
- `e'` is in normal form

Example:

```
    (1 + 2) + (4 + 5)
=~> 12
```

because

```
   (1 + 2) + (4 + 5)
=> 3       + (4 + 5)
=> 3       + 9
=> 12
```

and `12` is a *value* (normal form)

Theorems about Nano1

Let's prove something about Nano1!

1. Every Nano1 program terminates
2. Closed Nano1 programs don't get stuck
3. *Corollary (1 + 2):* Every closed Nano1 program evaluates to a value

How do we prove theorems about languages?

By induction.

Mathematical induction in PL

1. Induction on natural numbers

To prove $\forall n.P(n)$ we need to prove:

- *Base case:* $P(0)$
- *Inductive case:* $P(n+1)$ assuming the *induction hypothesis* (IH): that $P(n)$ holds

Compare with inductive definition for natural numbers:

```
data Nat = Zero     -- base case
         | Succ Nat -- inductive case
```

No reason why this would only work for natural numbers…

In fact we can do induction on *any* inductively defined mathematical object (= any datatype)!

- lists
- trees
- programs (terms)
- etc

2. Induction on terms

```
e ::= n | x
    | e1 + e2
    | let x = e1 in e2
```

To prove $\forall e.P(e)$ we need to prove:

- *Base case 1:* `P(n)`
- *Base case 2:* `P(x)`
- *Inductive case 1:* `P(e1 + e2)` assuming the IH: that `P(e1)` and `P(e2)` hold
- *Inductive case 2:* `P(let x = e1 in e2)` assuming the IH: that `P(e1)` and `P(e2)`hold

3. Induction on derivations

Our reduction relation `=>` is also defined *inductively*!

- Axioms are bases cases
- Rules with premises are inductive cases

To prove $\forall e, e'.P(e \Rightarrow e')$ we need to prove:

- *Base cases:* [`Add`], [`Let`]
- *Inductive cases:* [`Add-L`], [`Add-R`], [`Let-Def`] assuming the IH: that `P` holds of their premise

Theorem: Termination

Theorem I [Termination]: For any expression `e` there exists `e'` such that `e =~> e'`.

Proof idea: let's define the *size* of an expression such that

- size of each expression is positive
- each reduction step strictly decreases the size

Then the length of the execution sequence for `e` is *bounded* by the size of `e`!

```
size n                 = ???
size x                 = ???
size (e1 + e1)         = ???
size (let x = e1 in e2) = ???
```

Theorem: Termination

Term size:

```
size n                  = 1
size x                  = 1
size (e1 + e1)          = size e1 + size e2
size (let x = e1 in e2) = size e1 + size e2
```

Lemma 1: For any `e`, `size e > 0`.

Proof: By induction on the *term* `e`.

- *Base case 1:* `size n = 1 > 0`
- *Base case 2:* `size x = 1 > 0`
- *Inductive case 1:* `size (e1 + e2) = size e1 + size e2 > 0` because `size e1> 0` and `size e2 > 0` by IH.
- *Inductive case 2:* similar.

QED.

Theorem: Termination

Lemma 2: For any `e, e'` such that `e => e'`, `size e' < size e`.

Proof: By induction on the *derivation* of `e => e'`.

Base case `[Add]`.

- Given: the root of the derivation is

  `[Add]: n1 + n2 => n` where `n = n1 + n2`
- To prove: `size n < size (n1 + n2)`
- `size n = 1 < 2 = size (n1 + n2)`

Theorem: Termination

Lemma 2: For any `e, e'` such that `e => e'`, `size e' < size e`.

Inductive case `[Add-L]`.

- Given: the root of the derivation is `[Add-L]`:

```
     e1 => e1'
--------------------------
e1 + e2 => e1' + e2
```

- **To prove**: `size (e1' + e2) < size (e1 + e2)`
- **IH**: `size e1' < size e1`

```
     size (e1' + e2)
   = -- def. size
     size e1' + size e2
   < -- IH
     size e1 + size e2
   = -- def. size
     size (e1 + e2)
```

Inductive case `[Add-R]`. Try at home

Theorem: Termination

Lemma 2: For any `e, e'` such that `e => e'`, `size e' < size e`.

Base case `[Let]`.

- Given: the root of the derivation is `[Let]`: `let x = v in e2 => e2[x := v]`
- To prove: `size (e2[x := v]) < size (let x = v in e2)`

```
  size (e2[x := v])
= -- auxiliary lemma!
  size e2
< -- IH
  size v + size e2
= -- def. size
  size (let x = v in e2)
QED.
```

Inductive case `[Let-Def]`. Try at home

Nano2: adding functions

Syntax

We need to extend the syntax of expressions and values:

```
e ::= n | x              -- expressions
    | e1 + e2
    | let x = e1 in e2
    | \x -> e        -- abstraction
    | e1 e2          -- application

v ::= n                  -- values
    | \x -> e        -- abstraction
```

Operational semantics

We need to extend our reduction relation with rules for abstraction and application:

```
           e1 => e1'
[App-L] ----------------
        e1 e2 => e1' e2

          e => e'
[App-R] ------------
        v e => v e'

[App]   (\x -> e) v => e[x := v]
```

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Evaluation Order

```
   ((\x y -> x + y) 1) (1 + 2)
=> (\y -> 1 + y) (1 + 2)       -- [App-L], [App]
=> (\y -> 1 + y) 3             -- [App-R], [Add]
=> 1 + 3                       -- [App]
=> 4                           -- [Add]
```

Our rules define **call-by-value**:

1. Evaluate the function (to a lambda)
2. Evaluate the argument (to some value)
3. "Make the call": make a substitution of formal to actual in the body of the lambda

The alternative is **call-by-name**:

- do not evaluate the argument before "making the call"
- can we modify the application rules for Nano2 to make it call-by-name?

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Theorems about Nano2

Let's prove something about Nano2!

1. Every Nano2 program terminates (?)
2. Closed Nano2 programs don't get stuck (?)

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Theorems about Nano2

1. Every Nano2 program terminates (?)

   What about `(\x -> x x) (\x -> x x)`?

2. Closed Nano2 programs don't get stuck (?)

   What about `1 2`?

Both theorems are now false!

To recover these properties, we need to add *types*:

1. Every *well-typed* Nano2 program terminates

2. *Well-typed* Nano2 programs don't get stuck